THE UNIVERSAL PROPERTY OF EXTERNAL CHOICE

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1. Motivation

Given an $\mathcal{M}$-actegory $(\mathcal{X}, \bullet, \eta, \mu)$, for some monoidal $(\mathcal{M}, j, \odot)$, we can form the associated 'action bicategory', also known as $\mathbf{Para}_\bullet(\mathcal{X})$ [Cap+21a, Definition 2]. This has the same objects as $\mathcal{X}$ but morphisms $(M, f) : X \to Y$ are now given by a choice of scalar $M : \mathcal{M}$ and a morphism $f : M \bullet X \to Y$. Identities and composition are built using the unitor η and the multiplicator μ of $\bullet$:

$$\begin{aligned} 1_c &:= (j, \eta_c), \\ (M, f) \mathbin{⨟} (N, g) &:= (N \odot M, N \bullet f \mathbin{⨟} g). \end{aligned} \tag{1.1}$$

The bicategorical structure arises by using the morphisms of $\mathcal{M}$ as 2-cells. Explicitly, a 2-cell $(\alpha, =) : (M, f) \Rightarrow (N, g) : X \to Y$ is a morphism $\alpha : N \to M$ in $\mathcal{M}$ such that the triangle

$$\begin{array}{ccc} M \bullet X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \alpha \bullet X}\uparrow & & \\ N \bullet X & \xrightarrow[g]{} & Y \end{array} \tag{1.2}$$

commutes on the nose.[1]

The category $\mathbf{Para}_\bullet(\mathcal{X})$ is foundational in what we call 'categorical cybernetics' [Cru+21; Cap+21a; Cap+21b]. In particular, the 2-dimensional structure is surprisingly expressive and we keep finding amazing structures every time we decide to take it seriously. For instance, [Cap+21b] uses 2-cells to model 'players' in games; but also recently Gavranović and the author have uncovered a 'local adjunction' [MS89] between lenses and cartesian 2-optics, which are optics which 'take seriously' the second dimension (see [Bra+21]). Such an adjunction is described in [Gav22].

One of the main contributions of [Cap+21b] has been in extending the operator calculus for open games [Gha+16]. Most importantly it defines the so called 'external choice' operator, which is of fundamental importance for the definition of cybernetic systems, as confirmed by the recent work on open servers. So far, it has not been known what the universal property of such operator is, despite looking pretty natural to define.

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[1]If $\mathcal{X}$ were itself a bicategory, one could relax this definition and replace $(\alpha, =)$ with $(\alpha, \alpha^\sharp)$ where $\alpha^\sharp : \alpha \bullet X \mathbin{⨟} f \Rightarrow g$ is a 2-cell in $\mathcal{X}$.

2. External choice as a soft coproduct

Suppose $\mathcal{X}$ is a cartesian and cocartesian category, acting on itself by cartesian product. One can define the following morphism:

$$\delta_{M,N,X,X'} : (M \times N) \times (X + X') \longrightarrow M \times X + N \times X'. \tag{2.1}$$

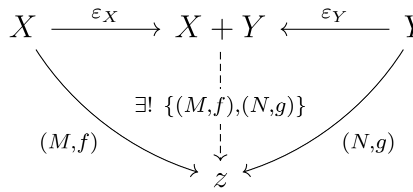
This in turn allows for the following definition:

Definition 2.1. Given $(M, f) : X \to Y$ and $(N, g) : X' \to Y'$ morphisms in $\mathbf{Para}_{\bullet}(\mathcal{X})$, their *external choice* is given by

$$(M, f) \,\&(N, g) := (M \times N,\ (M \times N) \times (X + X') \xrightarrow{\delta_{M,N,X,X'}} M \times X + N \times X' \xrightarrow{f+g} Y + Y'). \tag{2.2}$$

The name comes from the 'game semantics' of parametrised morphisms: in $(M, f) \,\&(N, g)$ the 'environment' chooses between X and X' while the 'agent' (which chooses parameters) will have to be ready to provide either a parameter M for f or a parameter N for g. Indeed, this is taken quite literally in [Cap+21b].

External choice is the closest thing to a coproduct we have in categories of parametrised morphisms, intuitively because $f + g$ does not have the form parametrised morphisms ought to have. More formally, if we try to define the coproduct in $\mathbf{Para}_{\times}(\mathcal{X})$, we are compelled to draw the following:



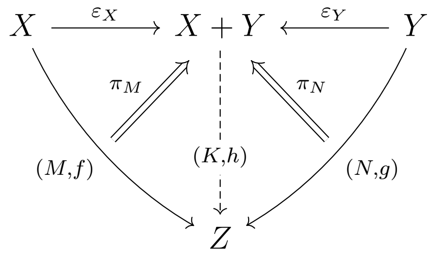
(2.3)

where $\{-, =\}$ denotes the universal copairing. We see that the choice of $X + Y$ and ε_X and ε_Y are fixed by the fact $\mathcal{X}$ is faithfully immersed in $\mathbf{Para}_{\times}(\mathcal{X})$, therefore a coproduct in the latter will be at least a coproduct in the first. We still have to define $\{(M, f), (N, g)\}$ though. Let's call the parameter of such a morphism K, and the underlying morphism h. To satisfy the required factorization condition, we need to have

$$\begin{aligned} K \times X \xrightarrow{\varepsilon_X} K \times (X + Y) \xrightarrow{h} Z &= M \times X \xrightarrow{f} Z \\ K \times Y \xrightarrow{\varepsilon_Y} K \times (X + Y) \xrightarrow{h} Z &= N \times Y \xrightarrow{g} Z \end{aligned} \tag{2.4}$$

Just focusing on the parameters, we see these equations are almost always mutually incompatible: they would force $K = M = N$!

The fact K should be M and N at the same time suggests that it should be something like $M \times N$, which is not *equal* to M or N but can produce either by projection. In other words, if we were to soften the equalities in (2.4) to directed morphisms, everything would go much more smoothly:



(2.5)

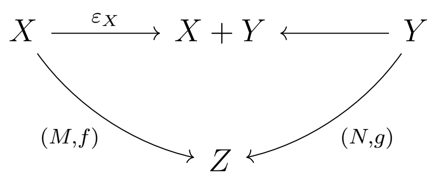
It's easy to see now that h is going to be equal to

$$M \times N \times (X+Y) \xrightarrow{\delta_{M,N,X,Y}} M \times X + M \times Y \xrightarrow{\{f,g\}} Z \tag{2.6}$$

where now $\{-,=\}$ denotes the universal copairing in $\mathcal{X}$.

However, the universal property of this objects is incomplete: it's not sufficient to have produced the morphism $(M \times N, h)$ and the 2-cells π_N, π_M, we want to show that these are universally chosen. In a sense, this is adding an extra layer of indirection to the usual universal property of coproducts.

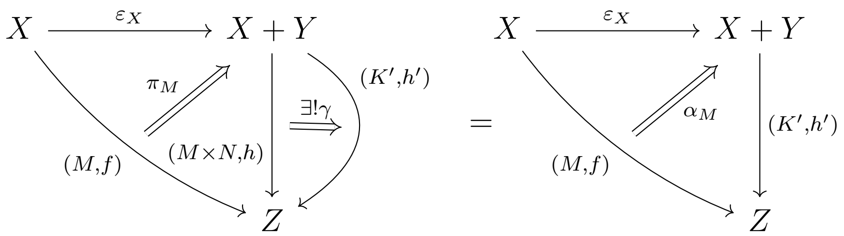
The universal property that seems natural to ask to $(M \times N, h)$ and π_N, π_M is to be universal 'fillers' of the incomplete diagram



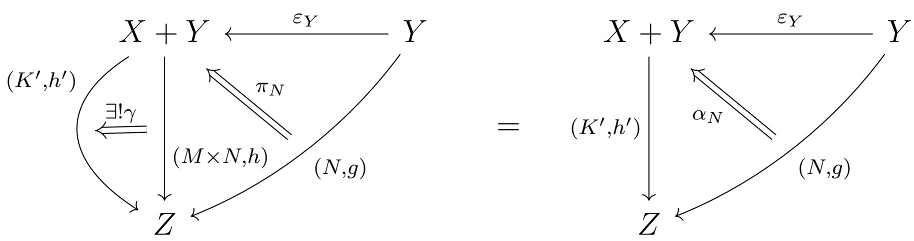
(2.7)

Put differently, we'd like $(M \times N, h)$ and π_M, π_N to be a sort of 'simultaneous Kan extension' of (M, f) and (N, g) along ε_X and ε_Y.

Explicitly, this means that for any morphism (K', h') and 2-cells $\alpha_M : (M, f) \Rightarrow \varepsilon_X \fatsemi (K', h')$ and $\alpha_N : (N, g) \Rightarrow \varepsilon_Y \fatsemi (K', h')$ filling the above diagram, there exist unique 2-cells $\gamma_M : (M \times N, h) \Rightarrow (K', h')$ such that:

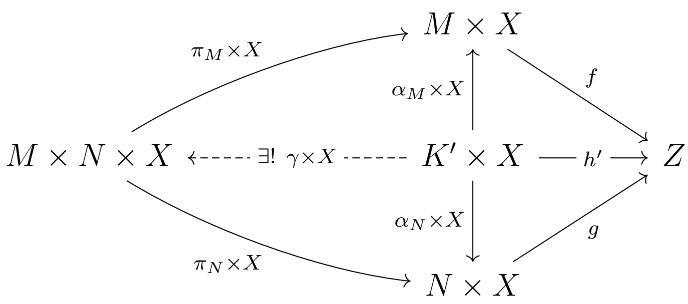


(2.8)



(2.9)

The 2-cells α_M and α_N would consist of morphisms $\alpha_M : K' \to M$ and $\alpha : K' \to N$ making the analogues of triangle (1.2) commute on the nose. We are then looking for a universal morphism $\gamma : K' \to M \times N$ that makes the following commute:



(2.10)

Therefore, at the end of the day, γ is given by universal pairing $\langle \alpha_M, \alpha_N \rangle$, showing that ultimately we are recycling the universal property of products in $\mathcal{X}$.

Remark 2.1.1. Reversing the 2-cells would produce a situation in which the universal property we end up using for K is that of the coproduct of M and N. The problem is, in general we don't have a morphism $(M + N) \times (X + Y) \to M \times X + M \times Y$ that we can use to define h, dooming our effort. A notable exception is when $\mathcal{X}$ is a category of pointed objects, in which case such a morphism is induced by precomposing δ with the morphism $M + N \to M \times N$ induced by the pointing.

We are gonna call this the **soft coproduct** of X and Y (as suggested by [Ker21]), though this (with reversed 2-cells) is called 'quasi-coproduct' in [Gra06, §1.7.3]. Moreover, we are gonna denote the 'soft universal morphism' (K, h) with the bracket notation, $\{(M, f), (N, g)\}_{\mathsf{soft}}$.

Observe now that the soft coproduct bracket $\{-, =\}_{\mathsf{soft}}$ is in the same relation with external choice of parametrised morphisms as the traditional universal copairing $\{-, =\}$ is with traditional coproduct of non-parameterised morphisms:

$$(M, f) \mathbin{\&} (N, g) = \{(M, f) \fatsemi \varepsilon_{X'}, (N, g) \fatsemi \varepsilon_{Y'}\}_{\mathsf{soft}}. \tag{2.11}$$

Indeed, when $M = N = 1$, π_M and π_N are identities thus recovering the usual coproduct diagram on the subcategory of non-parametrised morphisms.

Therefore, external choice is universally defined by soft coproducts in $\mathbf{Para}_\times(\mathcal{X})$.

Remark 2.1.2. The definition of soft coproduct might look quite unwieldy, but it can be stated much more cleanly in the language of local adjunctions, as done in [Gra06, Definition I.7.9.1]. The soft limit and colimit of a diagram $D : \mathbb{I} \to \mathbb{X}$ in a bicategory $\mathbb{X}$ are in fact given by the local left adjoint and local right adjoint, respectively, to the costant diagram functor $\Delta : \mathbb{X} \to 2\mathbb{C}\mathbf{at}(\mathbb{I}, \mathbb{X})$.

3. Conclusion

This note amplifies the underlying moral of [Gav22] (itself arising from Glaswegian folklore principles) that hint at the fact the mathematics of parametrised morphisms is intrinsecally higher categorical, and in a stronger sense that one might expect at first sight.

In fact, one has to *take bicategories seriously* and treat 2-cells on the same footing as 1-cells, and not resort to them just as a source of higher equality witnesses (thus morally treating 2-cells as undirected). If adjunctions and universal properties of 1-categories are so powerful and expressive, we should deploy them in the second dimension of bicategories as well. That's what soft limits and local adjunctions do.

This philosophy (and the tao permeating the above construction) might be also seen as answering an objection recently put forward by Smithe [Smi22] in noticing categories of parametrised morphisms don't seem to have enough limits or colimits. In the future, we'd like to expand this note in order to show that if we embrace the intrisic second dimension, such hurdles can be not only overcome, but turned into precious tools.

References

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